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A GENERAL MARTINGALE APPROACH TO DISCRETE TIME STOCHASTIC CONTROL AND ESTIMATION*

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Abstract

A general method of constructing system models for the solution of discrete time stochastic control and estimation problems is presented. The method involves the application of modern martingale theory and entails the judicious choice of certain sigma-algebras and martingales. General estimation equations are derived for observations taking values in a countable space, and previously obtained estimation equations are exhibited as special cases. Finally, an example of the application of these methods to a stochastic control problem is analyzed.

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I. Introduction

Much recent research has been directed toward the solution of estimation and stochastic control problems by means of the application of modern martingale theory [1]-[8]. In this paper we will be concerned with discrete time estimation and control problems in which the observations take values in a countable set. In particular, we will consider models of the form [11], [14]

$$x(t+1) = f_t(x^t, u(t)) + w(t+1) \tag{1}$$

$$z(t+1) = g_t(x^{t+1}, z^t, u(t)) + v(t+1) \tag{2}$$

where $x(t)$ is the state at time t, $x^t \triangleq \{x(0),\ldots,x(t)\}$, $z(t)$ is the observation at t, $z^t \triangleq \{z(0),\ldots,z(t)\}$, w and v are noise processes, and the control $u(t)$ is measurable with respect to σ-algebra $\sigma(z^t)$ generated by z^t. Segall [2] first derived discrete time estimation equations for 1-variate point process observations; however, he only allows f_t in (1) to depend on z^{t-1} (not on z^t, as we require here), and finds the one-step predictor $E[x(t)|z^{t-1}]$. Brémaud [4] has generalized Segall's work to k-variate point process observations.

In Section II we will demonstrate a method for constructing a system model leading to the solution of a given estimation or control problem with any prespecified (classical) information structure. This is accomplished by a judicious choice of sigma-algebras and martingales, and includes the information structures of Segall [2], Brémaud [4], Brémaud and Van Schuppen [5],[6], and our problem (1)-(2) as special cases. In Section III general estimation equations are derived for the models of Section II; it is assumed that the observations take values in a countable set. Vaca and Tretter [9] have derived analogous equations using an approach based on likelihood functions. As an example of (1)-(2), the control of a finite state Markov process is considered in Section IV; as in continuous time [3], the separation principle holds and a finite dimensional estimator is constructed.

II. Abstract System Model and Estimation Equation

Consider a discrete time observation process $y(t)$ which is modulated by the signal or state process $x(t)$, where $t \in \{0,1,\dots,T\}$. Assume both $x(t)$ and $y(t)$ are integrable processes. In this paper, we will restrict the value of the observation process $y(t)$ at time t to belong to a countable set $\{\rho_1,\dots,\rho_i,\dots\}$.

By using the martingale decomposition method, we can construct a general system model as follows. Let $\mathcal{G}_t$ and $\mathcal{B}_t$ be two increasing families of sigma-algebras such that $\mathcal{G}_t \supset \mathcal{B}_t \supset \mathcal{F}_{t'} \triangleq \sigma\{y(s), s \le t'\}$, where $t' \le t$; methods for judiciously choosing t', $\mathcal{G}_t$, and $\mathcal{B}_t$ for a particular problem will be indicated below. Also, assume $x(t)$ is $\mathcal{B}_t$-measurable. Let

$$z_i(t) \triangleq I(y(t)=\rho_i),$$

where I is the indicator function; thus $y(t) = \sum_{i=1}^{\infty} \rho_i z_i(t)$. Throughout this paper, the observation will be described equivalently by either $y(t)$ or $\{z_i(t), i=1,2,\dots\}$. We define, for $i=1,2,\dots$,

$$w(t+1) \triangleq x(t+1) - E^{\mathcal{B}_t}(x(t+1))$$

$$v_i(t+1) \triangleq z_i(t+1) - E^{\mathcal{G}_t}[z_i(t+1)],$$

where $E^{\mathcal{B}_t}[\cdot]$ denotes conditional expectation. Thus $w(t+1)$ and $v_i(t+1)$ are $\mathcal{B}_{t+1}$ and $\mathcal{G}_{t+1}$ martingale difference processes,[1] respectively [1], [2].

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[1] $m(t+1)$ is a $\mathcal{A}_{t+1}$ martingale difference process if $m(t+1)$ is measurable with respect to $\mathcal{A}_{t+1}$ and $E^{\mathcal{A}_t}(m(t+1)) = 0$. The reader is referred to [1] for more details.

In other words, we have the system model

$$x(t+1) = f_t + w(t+1) \tag{3}$$

$$z_i(t+1) = \mu_i(t+1) + v_i(t+1) \tag{4}$$

where $f_t \triangleq E^{\mathcal{B}_t}[x(t+1)]$ and $\mu_i(t+1) = E^{\mathcal{G}_t}[z_i(t+1)]$.

It is desired to compute the least square estimates of x(t+1) given the observations $\mathcal{F}_{t'+1}$; this estimate is the conditional mean $\hat{x}(t+1|t'+1) \triangleq E^{\mathcal{F}_{t'+1}}(x(t+1))$. Let

$$\begin{aligned}\bar{w}(t+1) &= \hat{x}(t+1|t'+1) - E^{\mathcal{F}_{t'}}(x(t+1)) \\ &= \hat{x}(t+1|t'+1) - \hat{f}_{t|t'}\end{aligned}$$

where $\hat{f}_{t|t'} \triangleq E^{\mathcal{F}_{t'}}(f_t)$. Then it is easy to see that $\bar{w}(t+1)$ is an $\mathcal{F}_{t'+1}$-martingale difference process. Similarly, let

$$\begin{aligned}\bar{v}_i(t'+1) &= z_i(t'+1) - E^{\mathcal{F}_{t'}}[z_i(t'+1)] \\ &= z_i(t'+1) - \hat{\mu}_i(t'+1)\end{aligned}$$

where $\hat{\mu}_i(t'+1) = E^{\mathcal{F}_{t'}}(\mu_i(t'+1))$. Also, $\bar{v}_i(t'+1)$ is an $\mathcal{F}_{t'+1}$-martingale difference process. Thus we have the abstract estimation equations

$$\hat{x}(t+1|t'+1) = \hat{f}_{t|t'} + \bar{w}(t+1) \tag{5}$$

$$z_i(t'+1) = \hat{\mu}_i(t'+1) + \bar{v}_i(t'+1), \tag{6}$$

$i = 1,2,...$

In the next section, we will determine $\bar{w}(t+1)$ in (5) in terms of the observation process (6) and $\mathcal{F}_{t'}$, thus generating a "recursive" estimation equation.

This model provides a general framework within which many problems can be analyzed by proper choice of $\mathcal{G}_t$, $\mathcal{B}_t$, and t'. For example, by choosing $\mathcal{G}_t = \sigma\{x^{t+1}, y^t\} = \sigma\{x^{t+1}, z^t\}$, $\mathcal{B}_t = \mathcal{G}_{t-1}$, and $t' = t-1$, we arrive at Segall's framework for deriving the one-step predictor [2]; notice from (3) that this yields a system equation of the form

$$x(t+1) = f_t(x^t, y^{t-1}) + w(t+1). \tag{7}$$

The model of Brémaud and Van Schuppen [5], [6] for deriving the one-step predictor is obtained by setting $\mathcal{B}_t = \mathcal{G}_{t-1}$ and $t' = t-1$. The model of (1)-(2) can be accommodated by defining $\mathcal{G}_t = \sigma\{x^{t+1}, y^t\}$ and $\mathcal{B}_t = \sigma\{x^t, y^t\}$, so that we have $f_t = f_t(x^t, y^t)$ and $g_t = g_t(x^{t+1}, y^t)$. The filtering and prediction problems for this model are derived by setting $t' = t$ and $t' < t$, respectively. Other problems, including the smoothing problem, can be approached similarly.

Estimation equations for special cases (such as those mentioned above) will follow immediately, by the proper choice of σ-algebras, from the results of the next section. Thus the need to rederive such estimation equations for each special case is eliminated.

III. General Estimation Equations

As noted before, we need to determine $\bar{w}(t+1)$ in (5) in terms of the observation process (6) and $\mathcal{F}_{t'}$. The general representation of the $\mathcal{F}_{t'+1}$-martingale difference $\bar{w}(t+1)$ is described in the following theorem. The proof is similar to that of Brémaud [4, p.36] for a special case.

<u>Theorem 1</u>: Consider the system model (3)-(4). If $y(t)$ takes values in a countable set $\{\rho_1, \rho_2, \ldots\}$, where we assume that each ρ_i occurs with non-zero probability so that $\hat{\mu}_i(t'+1) > 0$ for all i, then $\hat{x}(t+1|t'+1)$ can be expressed in the form

$$\hat{x}(t+1|t'+1) = \hat{f}_{t|t'} + \sum_{i=1}^{\infty} \left\{ \frac{1}{\hat{\mu}_i(t'+1)} E^{\mathcal{F}_{t'}}[(f_t + w(t+1))(z_i(t'+1) - \hat{\mu}_i(t'+1))] \cdot z_i(t'+1) \right\}$$

<u>Proof</u>: Notice that

$$\begin{aligned} \hat{x}(t+1|t'+1) - \hat{f}_{t|t'} &= E^{\mathcal{F}_{t'+1}}[x(t+1) - \hat{f}_{t|t'}] \\ &= E^{\mathcal{F}_{t'+1}}[f_t + w(t+1) - \hat{f}_{t|t'}] . \end{aligned}$$

Thus, we want to prove that, with probability one,

$$E^{\mathcal{F}_{t'+1}}[f_t + w(t+1) - \hat{f}_{t|t'}] = \sum_{i=1}^{\infty} \left\{ \frac{1}{\hat{\mu}_i(t'+1)} E^{\mathcal{F}_{t'}}[(f_t + w(t+1)) \cdot (z_i(t'+1) - \hat{\mu}_i(t'+1))] \right\} z_i(t'+1) \qquad (8)$$

By Lemma 10.1.3 of [13], (8) holds if and only if

$$E[(f_t + w(t+1) - \hat{f}_{t|t'})\, H_{t'+1}] = E\left\{\left[\sum_{i=1}^{\infty} \left\{\frac{1}{\hat{\mu}_i(t'+1)}\, E^{F_{t'}}[(f_t + w(t+1))\right.\right.\right.$$

$$\left.\left.\left.(z_i(t'+1) - \hat{\mu}_i(t'+1))]\right\} \cdot [z_i(t'+1)]\right] H_{t'+1}\right\} \tag{9}$$

for all $F_{t'+1}$ measurable random variables $H_{t'+1}$ which are almost surely bounded. Also, by Doob's representation theorem [16], we have

$$H_{t'+1} = H(t'+1, y(t'+1)) \tag{10}$$

where $H(t'+1,\rho_i)$, $i = 1,2,\ldots$, are $F_{t'+1}$-predictable ($E^{F_{t'}}(H(t'+1,\rho_i)) = H(t'+1,\rho_i)$), bounded processes.

After some calculations, the right-hand side of (9) becomes

$$E\left[\sum_{i=1}^{\infty}\left\{\frac{H(t'+1,\rho_i)}{\hat{\mu}_i(t'+1)}\, E^{F_{t'}}[(f_t + w(t+1) - \hat{f}_{t|t'})\, z_i(t'+1)]\, z_i(t'+1)\right\}\right]$$

By (10), we have that the left-hand side of (9) can be written

$$E[(f_t + w(t+1) - \hat{f}_{t|t'})\, H_{t'+1}] = E\left\{\sum_{i=1}^{\infty} (f_t + w(t+1) - \hat{f}_{t|t'})\, H(t'+1,\rho_i)\right.$$

$$\left.\cdot\, z_i(t'+1)\right\}$$

Thus, one needs only check the equality

$$E\left\{(f_t + w(t+1) - \hat{f}_{t|t'})\, H(t'+1,\rho_i)\, z_i(t'+1)\right\}$$

$$= E\left\{\frac{H(t'+1,\rho_i)}{\hat{\mu}_i(t'+1)}\, E^{F_{t'}}[(f_t + w(t+1) - \hat{f}_{t|t'})\, z_i(t'+1)]\, z_i(t'+1)\right\} \tag{11}$$

But the right-hand side of (11) is equal to (using predictability of H)

$$E\, E^{\mathcal{F}_{t'}}\left\{\frac{H(t'+1,\rho_i)}{\hat{\mu}_i(t'+1)}\, E^{\mathcal{F}_{t'}}[(f_t + w(t+1) - \hat{f}_{t|t'})\, z_i(t'+1)]\, z_i(t'+1)\right\}$$

$$= E\left\{H(t'+1,\rho_i)\, E^{\mathcal{F}_{t'}}[(f_t + w(t+1) - \hat{f}_{t|t'})\, z_i(t'+1)]\right\}$$

$$= E\, E^{\mathcal{F}_{t'}}[H(t'+1,\rho_i)(f_t + w(t+1) - \hat{f}_{t|t'})\, z_i(t'+1)],$$

which equals the left-hand side of (11). This proves the theorem.

The following corollaries specialize Theorem 1 by judicious choice of t' and the sigma-algebras $\mathcal{G}_t$ and $\mathcal{B}_t$.

Corollary 1. Under the assumptions of Theorem 1, if we set $\mathcal{B}_t = \sigma\{x^t, y^t\}$, $\mathcal{G}_t = \sigma\{x^{t+1}, y^t\}$, and $t' = t$, we have the filtering result

$$\hat{x}(t+1|t+1) = \hat{f}_{t|t} + \sum_{i=1}^{\infty}\left\{\frac{1}{\hat{\mu}_i(t+1)}\, E^{\mathcal{F}_t}[(f_t + w(t+1))(\mu_i(t+1) - \hat{\mu}_i(t+1))]\right\} \cdot z_i(t+1).$$

Corollary 2. Under the assumption of Corollary 1 but with $t' = t - 1$, we have the one-step predictor

$$\hat{x}(t+1|t) = \hat{f}_{t|t-1} + \sum_{i=1}^{\infty} \frac{1}{\hat{\mu}_i(t)}\, E^{\mathcal{F}_{t-1}}[(f_t + w(t+1))(z_i(t) - \hat{\mu}_i(t))]\, z_i(t).$$

If y(t) takes its values in a finite set $\{\rho_1,\ldots,\rho_m\}$ (i.e., $y(t) = \sum_{i=1}^{m} \rho_i z_i(t)$), then (as a special case of Theorem 1) $\hat{x}(t+1|t'+1)$ can be expressed in the form

$$\hat{x}(t+1|t'+1) = \hat{f}_{t|t'} + \sum_{i=1}^{m} \left\{ \frac{1}{\hat{\mu}_i(t'+1)} E^{F_{t'}} [(f_t + w(t+1))(z_i(t'+1) - \hat{\mu}_i(t'+1))] \cdot z_i(t'+1) \right\}. \quad (12)$$

However, an alternative proof which is of independent interest is also presented here; the proof is based upon the martingale representation theorem [5].

<u>Corollary 3</u>. Under the assumptions of Theorem 1, if $y(t) \varepsilon \{\rho_1,\ldots,\rho_m\}$, then $\hat{x}(t+1|t'+1)$ can be expressed as in (12).

<u>Proof</u>: Assume the signal process $x(t)$ is scalar. By the martingale representation theorem and the innovations theorem [5], there exist $F_{t'+1}$ predictable processes $K_i(t'+1)$, $i=1,\ldots,m$, such that

$$\bar{w}(t+1) = \hat{x}(t+1|t'+1) - \hat{f}_{t|t'} = \sum_{i=1}^{m} K_i(t'+1) [z_i(t'+1) - \hat{\mu}_i(t'+1)]$$

since $\bar{w}(t+1)$ is an $F_{t'+1}$ martingale difference process. Note that

$$\sum_{i=1}^{m} z_i(t'+1) = 1\,, \quad (13)$$

$$\sum_{i=1}^{m} \hat{\mu}_i(t'+1) = 1\,. \quad (14)$$

Now,

$$\sum_{i=1}^{m} K_i(t'+1)\ [z_i(t'+1) - \hat{\mu}_i(t'+1)]$$

$$= \sum_{i=1}^{m-1} K_i(t'+1)\ [z_i(t'+1) - \hat{\mu}_i(t'+1)] - \sum_{i=1}^{m-1} K_m(t'+1)\ [z_i(t'+1) - \hat{\mu}_i(t'+1)]$$

$$= \sum_{i=1}^{m-1} (K_i(t'+1) - K_m(t'+1))\ [z_i(t'+1) - \hat{\mu}_i(t'+1)]$$

The first equality above is obtained by using (13) and (14).

Let $L_i(t'+1) = K_i(t'+1) - K_m(t'+1)$. It is easy to see that $L_i(t'+1)$ is an $F_{t'+1}$ predictable process. Thus, we have

$$\hat{x}(t+1|t'+1) - \hat{f}_{t|t'} = \sum_{i=1}^{m-1} L_i(t'+1)\ [z_i(t'+1) - \hat{\mu}_i(t'+1)] \tag{15}$$

Multiplying both sides by $z_i(t'+1) - \hat{\mu}_i(t'+1)$ for $i = 1,\ldots,m-1$, then taking conditional expectations with respect to $F_{t'}$, we have

$$\begin{bmatrix} E^{F_{t'}}[(f_t + w(t+1))(z_1(t'+1) - \hat{\mu}_1(t'+1))] \\ \vdots \\ E^{F_{t'}}[(f_t + w(t+1))(z_{m-1}(t'+1) - \hat{\mu}_{m-1}(t'+1))] \end{bmatrix}$$

$$= \begin{bmatrix} \hat{\mu}_1(t'+1) - \hat{\mu}_1^2(t'+1) & -\hat{\mu}_1(t'+1)\,\hat{\mu}_2(t'+1) & & -\hat{\mu}_1(t'+1)\hat{\mu}_{m-1}(t'+1) \\ -\hat{\mu}_2(t'+1)\,\hat{\mu}_1(t'+1) & \hat{\mu}_2(t'+1) - \hat{\mu}_2^2(t'+1) & & -\hat{\mu}_2(t'+1)\hat{\mu}_{m-1}(t'+1) \\ \vdots & & \ddots & \vdots \\ -\hat{\mu}_{m-1}(t'+1)\hat{\mu}_1(t'+1) & \cdots & \cdots & \hat{\mu}_{m-1}(t'+1) - \hat{\mu}_{m-1}^2(t'+1) \end{bmatrix}$$

$$\cdot \begin{bmatrix} L_1(t'+1) \\ \vdots \\ L_{m-1}(t'+1) \end{bmatrix} \triangleq A \cdot \begin{bmatrix} L_1(t'+1) \\ \vdots \\ L_{m-1}(t'+1) \end{bmatrix}$$

It can easily be shown that A is a positive definite matrix, so we can find a unique solution for $L_i(t'+1)$, $i = 1, \ldots, m-1$. By some calculations, we have

$$L_i(t'+1) = \frac{1}{\hat{\mu}_i(t'+1)} E^{F_{t'}}[(f_t + w(t+1))(z_i(t'+1) - \hat{\mu}_i(t'+1))]$$

$$- \frac{1}{\hat{\mu}_m(t'+1)} E^{F_{t'}}[(f_t + w(t+1))(z_i(t'+1) - \hat{\mu}_i(t'+1))]$$

Substituting into (15), we have

$$\hat{x}(t+1|t'+1) - \hat{f}_{t|t'} = \left[\sum_{i=1}^{m-1} \left\{ \frac{1}{\hat{\mu}_i(t'+1)} E^{F_{t'}} [(f_t + w(t+1))(z_i(t'+1) - \hat{\mu}_i(t'+1))] \right\} \right.$$

$$\left. + \frac{1}{\hat{\mu}_m(t'+1)} E^{F_{t'}} [(f_t + w(t+1))(z_m(t'+1) - \hat{\mu}_m(t'+1))] \right]$$

$$\cdot (z_i(t'+1) - \hat{\mu}_i(t'+1))$$

$$= \sum_{i=1}^{m} \left\{ \frac{1}{\hat{\mu}_i(t'+1)} E^{F_{t'}} [(f_t + w(t+1))(z_i(t'+1) - \hat{\mu}_i(t'+1))] \right\}$$

$$\cdot (z_i(t'+1) - \hat{\mu}_i(t'+1))$$

$$= \sum_{i=1}^{m} \left\{ \frac{1}{\hat{\mu}_i(t'+1)} E^{F_{t'}} [(f_t + w(t+1))(z_i(t'+1) - \hat{\mu}_i(t'+1))] \right\}$$

$$\cdot z_i(t'+1).$$

The second equality is obtained via (13) and (14). After expanding the second equality and interchanging the summation and conditional expectation, the last equality is also obtained by (13) and (14). Thus, the theorem is proved for scalar $x(t)$. Obviously, we can generalize to vector processes $x(t)$ by using each component of $x(t)$ in the above proof. Then, we combine all the components to get the result, which remains the same in the vector case.

Theorem 2: Under the assumptions of Theorem 1, the smoothed estimate satisfies

$$x(t|\bar{t}) = \hat{x}(t|t) + \sum_{s=t}^{\bar{t}-1} \sum_{i=1}^{\infty} \left\{ \frac{1}{\hat{\mu}_i(s+1)} E^{F_s}[(f_t + w(t+1))(z_i(s+1) - \hat{\mu}_i(s+1))] \right\} z_i(s+1) \tag{16}$$

for $\bar{t} > t$.

Proof: Let

$$\eta(s+1) = E^{F_{s+1}}(x(t)) - E^{F_s}(x(t)) \tag{17}$$

for $s \geq t$. Then it is easy to prove that $\eta(s+1)$ is an F_{s+1}-martingale difference process. By summing the terms (17) from $s = t$ to $s = \bar{t} - 1$, we have

$$\hat{x}(t|\bar{t}) = \hat{x}(t|t) + \sum_{s=t}^{\bar{t}-1} \eta(s+1).$$

By using a proof similar to that of Theorem 1, it can be shown that

$$\eta(s+1) = \sum_{i=1}^{\infty} \frac{1}{\hat{\mu}_i(s+1)} E^{F_s}[(f_t + w(t+1))(z_i(s+1) - \hat{\mu}_i(s+1))] \, z_i(s+1)$$

Thus, we have the result (16).

Remarks: (i) The form of the estimation equations obtained by Segall [2] for a special case follows from (12) by a simple calculation.

(ii) Results analogous to Corollaries 1 and 2 and Segall's results [2] can also be derived for the smoothed estimate. In addition, estimation equations for other information structures can be derived by judicious choice of G_t, B_t, and t' (or $\bar{t}$ in smoothing problems), as discussed at the end of Section II.

(iii) In general, the optimal nonlinear estimator is infinite dimensional; in other words, $\hat{x}(t+1|t'+1)$ will depend on all higher order conditional moments. Representation results for higher order conditional moments can also be derived by using the methods described above.

(iv) In particular, in Corollary 1, $\hat{f}_{t|t} = E^{F_t}[f_t]$ is actually the one step predicted estimate $\hat{x}(t+1|t)$. Since f_t is a function of x^t and y^t, $\hat{x}(t+1|t)$ cannot be computed simply in terms of $\hat{x}(t|t)$, but in fact depends in general on all of the conditional probability distributions $P(x(s)|F_t)$, $s \leq t$. However, a special case which is given in the next section will result in a finite dimensional recursive estimator.

IV. Example -- Stochastic Control Problem

In this section, we present an example illustrating the application of the techniques in Section III to a class of stochastic control problems of the form (1)-(2). For these problems the optimal filter is finite dimensional and the separation principle [10] holds (that is, the optimal control u(t) is only a function of the filtered estimate $\hat{\alpha}(t|t)$). Thus, we can see the strong relationship between the estimation and stochastic control problems. This example is motivated by the corresponding 1-variate continuous time problem [3] and the 1-variate discrete time prediction example of [2]. For a different approach, see [12].

Consider a stochastic control problem in which the signal is a controlled finite state Markov chain x(t) such that $x(t) \varepsilon \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$. Let the initial probabilities be

$$\pi_0^k = P\{x(0) = \gamma_k\}, \qquad k = 1,2,\ldots,n,$$

and transition probabilities be

$$\pi_t^{kj}(u(t)) = P\{x(t+1) = \gamma_j \mid x(t) = \gamma_k, u(t)\}$$

where u(t) is the control at time t. Define, as in [2], [3],

$$\alpha_j(t) = \begin{cases} 1 & \text{if } x(t) = \gamma_j \\ 0 & \text{otherwise} \end{cases}$$

and the n-vector

$$\alpha(t) = [\alpha_1(t) \ldots \alpha_n(t)]^T$$

and the n×n matrix

$$Q_t(u(t)) = [\pi_t^{kj}(u(t))].$$

The observation process y(t) is assumed to take values in a countable set $\{\rho_1, \rho_2, \ldots, \rho_m\}$ and the equivalent point processes $z_i(t)$ are defined as in Section II. As in Corollaries 1-2, we let $t' = t$, $\mathcal{G}_t(u) = \sigma\{x^{t+1}, y^t\}$, $\mathcal{B}_t(u) = \sigma\{x^t, y^t\}$, and $\mathcal{F}_t(u) = \sigma\{y^t\}$, where u is the control law. The admissible control laws are such that u(t) is $\mathcal{F}_t(u)$ measurable, and an optimal control is one that optimizes the performance index

$$J(u) = E\{L_T(x(T)) + \sum_{t=0}^{T-1} L_t(x(t), u(t))\},$$

where $L_t(x(t), u(t))$, $t=0, \ldots, T$, are measurable functions.

We first derive a system model of the form (3)-(4) with the relevant choice of σ-algebras. Let

$$w_j(t+1) = \alpha_j(t+1) - \sum_{k=1}^{n} \pi_t^{kj}(u(t))\, \alpha_k(t);$$

then $w_j(t+1)$ is a $\mathcal{B}_{t+1}(u)$-martingale difference process, since

$$E^{\mathcal{B}_t}[w_j(t+1)] = E^{\mathcal{B}_t}[\alpha_j(t+1)] - E^{\mathcal{B}_t}[\sum_{k=1}^{n} \pi_t^{kj}(u(t))\, \alpha_k(t)]$$

$$= \sum_{k=1}^{n} \pi_t^{kj}(u(t))\, \alpha_k(t) - \sum_{k=1}^{n} \pi_t^{kj}(u(t))\, \alpha_k(t) = 0$$

In vector form, the signal model thus becomes

$$\alpha(t+1) = Q_t^T(u(t))\alpha(t) + w(t+1). \tag{18}$$

In an analogous manner to [2],

$$ {}_i s_t^{kj}(u(t-1)) = P\{x(t+1)=\gamma_j,\ y(t)=\rho_i \mid x(t)=\gamma_k, u(t-1)\} $$

and define the n×n matrix

$$ {}_i S_t(u(t-1)) = [\,{}_i s_t^{kj}(u(t-1))\,]. $$

The observations are given by (4), where

$$ \mu_i(t+1) \triangleq E^{G_t}[z_i(t+1)] = \sum_{j=1}^{n} \sum_{k=1}^{n} {}_i s_{t+1}^{kj}(u(t))\ \alpha_k(t+1) $$

$$ = I_{1\times n}\, {}_i S_{t+1}^T (u(t))\ \alpha(t+1) \tag{19} $$

where $I_{1\times n} = [\,1 \ldots . 1\,]$ is a n-row vector. Now we substitute (18) and (19) into the filtering equation (12) and obtain a finite dimensional filter as follows:

$$ \hat{\alpha}(t+1|t+1) = Q_t^T(u(t))\ \hat{\alpha}(t|t) + \sum_{i=1}^{m} \left\{ \frac{1}{\hat{\mu}_i(t+1)} \right. $$

$$ \cdot \left[\mathrm{diag}\left(\sum_{k=1}^{n} \pi_t^{ki}(u(t))\hat{\alpha}_k(t|t) \right) {}_i S_{t+1}(u(t))\ I_{n\times 1} \right. $$

$$ \left. \left. - Q_t^T(u(t))\ \hat{\alpha}(t|t)\ \hat{\mu}_i(t+1) \right] \right\} z_i(t+1) \tag{20} $$

where $\hat{\mu}_i(t+1) = I_{1\times n}\ {}_i S_{t+1}^T(u(t))\, Q_t^T(u(t))\ \hat{\alpha}(t|t)$ and $\mathrm{diag}(\beta^i)$ denotes a diagonal n×n matrix with the diagonal element β^i. The initial condition is $\hat{\alpha}(0|0) = E\{\alpha(0)\} = [\,\pi_0^1 \ \ldots .\ \pi_0^n\,]^T$.

Since

$$ x(t+1) = [\,\gamma_1 \ldots .\ \gamma_n\,]\,\alpha(t+1)\ , $$

we have

$$\hat{x}(t+1|t+1) = [\gamma_1 \; \ldots . \; \gamma_n] \; \hat{\alpha}(t+1|t+1) .$$

The performance index is

$$J(u) = E\{L_T(x(T)) + \sum_{t=0}^{T-1} L_t(x(t),u(t))\}$$

$$= E\{[L_T(\gamma_1) \ldots L_T(\gamma_n)] \; \alpha(T) + \sum_{t=0}^{T-1} [L_t(\gamma_1,u(t)) \ldots L_t(\gamma_n,u(t))] \, \alpha(t)\}$$

$$= E\{\tilde{L}_T \; \hat{\alpha}(T|T) + \sum_{t=0}^{T-1} \tilde{L}_t(u(t)) \; \hat{\alpha}(t|t)\} , \tag{21}$$

where $\tilde{L}_T$, $\tilde{L}_t$ are vectors defined in the obvious manner. Thus, the separation principle holds in this case since we have perfect observations of $\hat{\alpha}(t|t)$ for all t; that is, there is no loss of generality in considering only control laws of the form $u(t) = u(\hat{\alpha}(t|t))$. Hence if the dynamic programming algorithm for the perfect observation problem (20)-(21) has a solution, then the optimal feedback control u(t) for the original problem will only depend on $F_t(u)$ via $\hat{\alpha}(t|t)$, which evolves according to the finite dimensional filtering equation (20).

V. Conclusion

A more general method of constructing system models for the solution of discrete time stochastic control and estimation problems has been presented. It involves the judicious choice of certain σ-algebras generated by the signal and observation processes. General estimation equations have been derived, and numerous special cases have been considered. The relationship between these estimation results and stochastic control problems has been demonstrated by the example of finite state Markov signals, for which the separation principle holds and a finite dimensional filter exists. The representation theorem, and hence the estimation equations, become much more difficult to interpret if the observations take values in an uncountable space [5], [6], and explicit results are difficult to obtain. However, explicit finite dimensional estimators for a class of nonlinear systems with additive Gaussian observation noise have been derived in [15].

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A general method of constructing system models for the solution of discrete time stochastic control and estimation problems is presented. The method involves the application of modern martingale theory and entails the judicious choice of certain sigma-algebras and martingales. General estimation equations are derived for observations taking values in a countable space, and previously obtained estimation equations are exhibited as special cases. Finally, an example of the application of these methods to a stochastic control problem is analyzed.

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